Non-linear σ-model:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\sigma\,\partial^\mu\sigma - 2\sigma^2\vec{D}_\mu\cdot\vec{D}^\mu - \frac{1}{2}\mu^2\sigma^2 - \frac{\lambda}{4}\sigma^4, \quad (1)$$

where $\vec{D}_\mu \equiv \dfrac{\partial_\mu\vec{\zeta}}{1+\vec{\zeta}^2}$

$\longrightarrow$ still invariant under SO(4):

- under isospin trfs. with infinitesimal $\vec{\theta}$,

$$\delta\vec{\zeta} = \vec{\theta}\times\vec{\zeta}, \qquad \delta\sigma = 0 \quad (2)$$

and $\mathcal{L}$ is $SU(2)_{iso}$-invariant

- under broken symmetry trfs. $SU(2)_{chir}$

$$\delta\vec{\phi} = 2\vec{\varepsilon}\,\phi_4, \quad \delta\phi_4 = -2\vec{\varepsilon}\cdot\vec{\phi}$$

then from $\zeta_a \equiv \dfrac{\phi_a}{\phi_4+\sigma}$ we get

$$1-\vec{\zeta}^2 = \frac{(\phi_4+\sigma)^2-\vec{\phi}^2}{(\phi_4+\sigma)^2} = \frac{2\phi_4^2+2\phi_4\sigma}{(\phi_4+\sigma)^2}$$

$$= \frac{2\phi_4}{\phi_4+\sigma}$$

$$\longrightarrow \delta\vec{\zeta} = \vec{\varepsilon}(1-\vec{\zeta}^2) + 2\vec{\zeta}(\vec{\varepsilon}\cdot\vec{\zeta}), \quad \delta\sigma = 0 \quad (3)$$

and thus $\delta\vec{D}_\mu = 2(\vec{\zeta}\times\vec{\varepsilon})\times\vec{D}_\mu$ is a linear (though field-dependent) isospin rotation

$\longrightarrow$ $\mathcal{L}$ remains invariant!

The trf. rules (2) and (3) specify a "non-linear realization" of $SU(2) \times SU(2)$

Passing to the limit

$$\mu, \lambda \longrightarrow \infty$$

with $\frac{|\mu|}{\sqrt{\lambda}} = \langle \sigma \rangle$ constant,

σ can be integrated out, i.e. set to its expectation value

$$\rightarrow \quad \mathcal{L} = -\frac{F^2}{2} \vec{D}_\mu \cdot \vec{D}^\mu = -\frac{F^2}{2} \frac{\partial_\mu \vec{\xi} \cdot \partial^\mu \vec{\xi}}{2(1+\vec{\xi}^2)^2}$$

where $F = 2\langle \sigma \rangle$

Choosing normalization

$$\vec{\pi} = F \vec{\xi}$$

we get

$$\mathcal{L} = -\frac{1}{2} \frac{\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi}}{(1 + \vec{\pi}^2/F^2)^2} \qquad (4)$$

$\rightarrow$ $\frac{1}{F}$ acts as coupling parameter accompanying interaction of each additional pion

(4) specifies a "non-linear σ-model"

Remark:

The non-linear σ-model has the following geometric interpretation:

$$
\begin{array}{l}
G \text{ (global symmetry group)} \\
\big\downarrow \text{ spontaneous symmetry breaking} \\
H \subset G \text{ (unbroken group)}
\end{array}
$$

$\rightarrow$ Goldstone bosons parametrize the space G/H as a manifold

Speify to $G = O(N+1)$, $H = O(N)$

$\rightarrow$ $O(N+1)/O(N) \sim S^N$

in our case $G = SO(4) \sim SU(2) \times SU(2)$ and $H = SU(2)$

$\rightarrow$ $G/H \sim (SU(2) \times SU(2))/SU(2) \simeq SU(2) \simeq S^3$

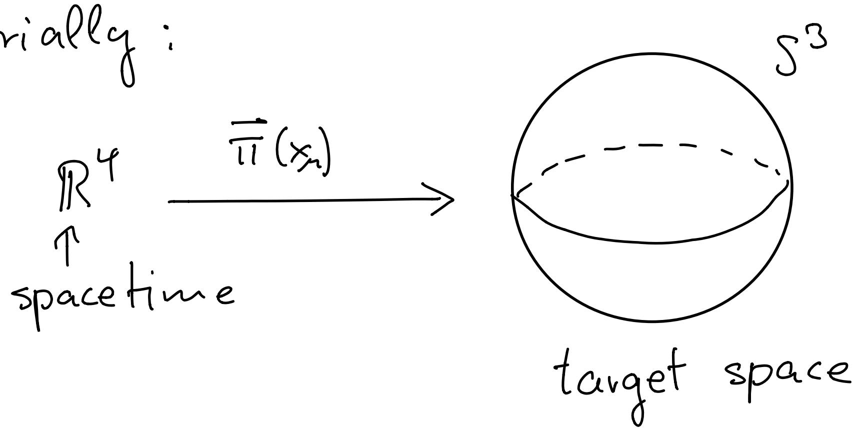
Now S^3 has $SO(3)$-invariant metric:

$$
ds^2 = \frac{(d\vec{x})^2}{(1 + |\vec{x}|^2)^2}
$$

with north pole at $x = \infty$.

This our $\vec{\pi}$-Lagrangian (4) !

pictorially:



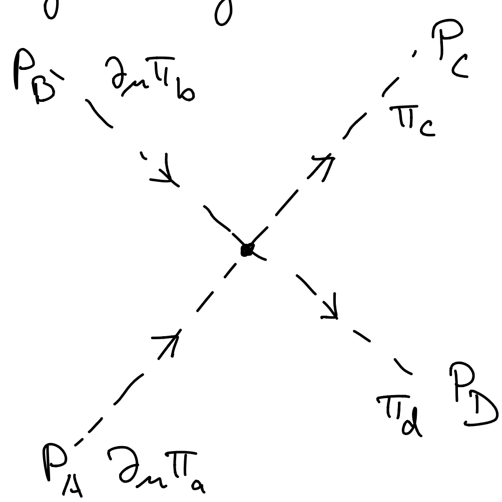
Now let's continue our discussion of pion interactions

$\longrightarrow$ expand $\mathcal{L}$ in powers of $\frac{1}{F}$:

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \vec{\pi}\right)\cdot\left(\partial^\mu \vec{\pi}\right) + \frac{1}{2F^2}\vec{\pi}^{\,2}\left(\partial_\mu \vec{\pi}\right)^2 - \frac{1}{2F^4}\vec{\pi}^{\,4}\left(\partial_\mu \vec{\pi}\right)^2$$

$$+ \dots$$

$\pi\pi$-scattering to lowest order in $\frac{1}{F}$ is then given by



$$S = i(2\pi)^4 \delta^4(p_A + p_B - p_C - p_D) \times \mathcal{M}(2\pi)^{-6}(16 E_A E_B E_C E_D)^{-1/2}$$

$p \sim Q \ll F$

energy of pions

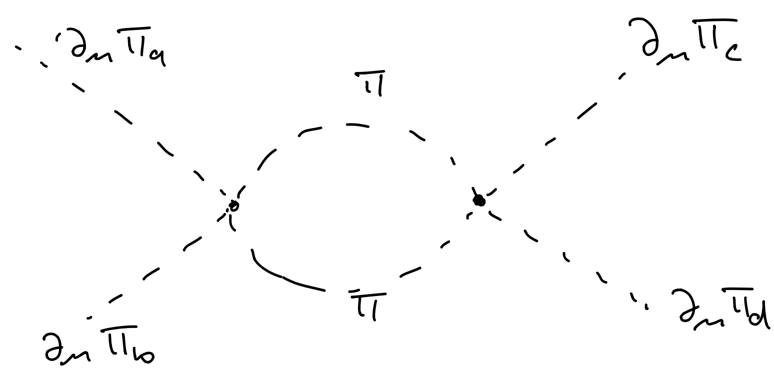
where

$$M^{(\nu=2)}_{abcd} = 4F^{-2}\Big[\delta_{ab}\,\delta_{cd}\left(-p_A\cdot p_B - p_C\cdot p_D\right)$$
$$+\,\delta_{ac}\,\delta_{bd}\left(p_A\cdot p_C + p_B\cdot p_D\right) + \delta_{ad}\,\delta_{bc}\left(p_A\cdot p_D + p_B\cdot p_C\right)\Big],$$

where a,b,c,d are isovector indices

ν indicated number of derivatives

At one-loop we get



→ gives corrections to Lagrangian of the form $(\vec{D}_\mu\cdot\vec{D}_\mu)^2$ and $(\vec{D}_\mu\cdot\vec{D}_\nu)(\vec{D}^\mu\cdot\vec{D}^\nu)$

($\mathcal{L}$ is non-renormalizable but gives sensible results if we take <u>all</u> counter-terms into account)

→ most general effective Lagrangian consistent with our symmetries:

$$\mathcal{L}_{eff} = -\frac{F^2}{2}\vec{D}_\mu\cdot\vec{D}^\mu - \frac{c_4}{4}\left(\vec{D}_\mu\cdot\vec{D}^\mu\right)^2 - \frac{c_4'}{4}\left(\vec{D}_\mu\cdot\vec{D}_\nu\right)\left(\vec{D}^\mu\cdot\vec{D}^\nu\right)$$
$$- \ldots$$

- each derivative in each interaction vertex $\longrightarrow Q$
- internal pion propagator $\longrightarrow Q^{-2}$
- d^4q associated to each loop $\rightarrow Q^4$

$\rightarrow$ general connected diagram of order Q^ν:

$$\nu = \sum_i V_i d_i - 2I + 4L$$

where d_i is number of derivatives in an interaction of type i, V_i is # vertices of type i, I is # pion lines, L is # loops

We have:

$$L = I - \sum_i V_i + 1$$

$$\rightarrow \nu = \sum_i V_i (d_i - 2) + 2L + 2$$

Observe: $\nu \geq 2$ $\quad (d_i \geq 2, \; L \geq 0)$

$\nu = 2$ ✓

A $\nu = 4$ we get:

$$M^{(\nu=4)}_{abcd} = \frac{\delta_{ab}\delta_{cd}}{F^4}\Big[-\frac{1}{2\pi^2} s^2 \ln(-s) - \frac{1}{12\pi^2}(u^2 - s^2 + 3t^2)\ln(-t)$$
$$- \frac{1}{12\pi^2}(t^2 - s^2 + 3u^2)\ln(-u) + \frac{1}{3\pi^2}(s^2 + t^2 + u^2)\ln(\Lambda^2)$$
$$- \frac{1}{2} c_4 s^2 - \frac{1}{4} c_4' (t^2 + u^2)\Big] + \text{crossed terms}$$

where

$$s = -(p_A + p_B)^2, \quad t = -(p_A - p_C)^2, \quad u = -(p_A - p_D)^2$$

→ define renormalized couplings:

$$c_{4R} = c_4 - \frac{2}{3\pi^2} \ln\left(\frac{\Lambda^2}{\mu^2}\right),$$

$$c'_{4R} = c'_4 - \frac{4}{3\pi^2} \ln\left(\frac{\Lambda^2}{\mu^2}\right),$$

where μ is renormalization scale of order Q

→ $\mathcal{M}^{(\nu=4)}_{abcd}$

$$= \frac{\delta_{ab}\delta_{cd}}{F^4}\Big[-\frac{1}{2\pi^2} s^2 \ln\left(\frac{-s}{\mu^2}\right) - \frac{1}{12\pi^2}(u^2 - s^2 + 3t^2)\ln\left(\frac{-t}{\mu^2}\right)$$

$$- \frac{1}{12\pi^2}(t^2 - s^2 + 3u^2)\ln\left(\frac{-u}{\mu^2}\right) - \frac{1}{2} c_{4R}\, s^2 - \frac{1}{4} c'_{4R}(t^2 + u^2)\Big]$$

+ crossed terms.

<u>Axial-vector current:</u>

$$\vec{\varepsilon} \cdot \vec{A}^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \vec{\zeta})} \cdot \delta\vec{\zeta} \qquad \text{(Noether's theorem)}$$

$$\rightarrow \vec{A}^\mu = -(1 - \vec{\zeta}^2)\frac{\partial \mathcal{L}}{\partial(\partial_\mu \vec{\zeta})} - 2\vec{\zeta}\,\vec{\zeta}\cdot\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \vec{\zeta})}\right)$$

is current generated by $2\vec{x} = 2\gamma_5 \vec{t} = \gamma_5 \vec{\tau}$

we find:

$$\vec{A}^\mu = F\left[\partial^\mu \vec{\pi}\,\frac{(1 - \vec{\pi}^2/F^2)}{(1 + \vec{\pi}^2/F^2)} + \frac{2\vec{\pi}\,(\vec{\pi}\cdot\partial^\mu\vec{\pi})}{F^2(1 + \vec{\pi}^2/F^2)^2}\right] + \ldots$$

→ in lowest order the pion decay amplitude $\langle VAC | \vec{A}^\mu | \pi \rangle$ is

$$\langle VAC | F \partial^\mu \pi_a(x) | \pi_b \rangle + O(Q^2/F^2)$$

$$= F \partial^\mu \underbrace{\langle VAC | \pi_a(x) | \pi_b \rangle}_{= \frac{e^{+ip_{\pi_b} \cdot x} \delta_{ab}}{(2\pi)^{3/2} \sqrt{2p^0_{\pi_b}}}} + O(Q^2/F^2)$$

$$= i \frac{F p^\mu_{\pi_b} e^{ip_{\pi_b} \cdot x} \delta_{ab}}{(2\pi)^{3/2} \sqrt{2p^0_{\pi_b}}} + O(Q^2/F^2) \ , \quad p_{\pi_b} \sim Q$$

$$\rightarrow \quad F_\pi = F \ !$$

→ Lorentz invariance tells us that higher order corrections must be proportional to powers of p_π^2 / F_π^2 (small as $m_\pi \sim 0$ approximately)